ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 6, Issue 3, pp: (528-539), Month: July - September 2018, Available at: www.researchpublish.com

An Analysis of a Two-State Retrial Queueing Model with Balking

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Abstract: **In this paper, we have proposed a single server retrial queueing system with balking. At the arrival time, if the server is busy then the arriving job joins the orbit or balks the system. On the other hand, if the server is free then the service of arriving job gets started immediately. The primary and repeating customers follow Poisson distribution. Service times are exponentially distributed. Solving the difference - differential equations recursively, the time dependent probabilities of exact number of arrivals and exact number of departures at when the server is free or when the server is busy from the system are obtained. Some interesting system performance measures are also obtained. The effect of various parameters on our model are illustrated numerically and presented graphically.**

Keywords: **Retrial, Arrivals, Departures, Queueing, Probability, Balking.**

1. INTRODUCTION

Recently there have been significant contributions to retrial queueing system. In this system, arriving customer who finds the server busy upon arrival is forced to leave the service area and repeat his demand after some random amount of time. In between these trials, a blocked customer who remains in a retrial group is said to be in orbit. In queues, the customers are allowed to conduct retrials that have applications in telecommunications networks. There are plenty of literatures available on the retrial queues. For a detailed analysis of main results and bibliographical information about retrial queues, refereed the works of Falin and Templeton [6], Aissani [3] and Atralejo [1, 2] etc.

On arrival if a customer finds the expected waiting time in the system seems to be more than his available time, then he refuses to join the system. In this situation the system is said to be balked. Haight [8] was first to work on the concept of balking. A very few works have been done with the concept of retrial queueing system with balking. Such retrial queueing models apply in many real world situations like call centers, web access and computer systems etc. $M^{[x]}/G/1$ queue with variant vacations and balking was studied by Ke [9]. Wang and Li [10, 11] and Baruah et al. [14] also discussed the concept of balking.

Pegden and Rosenshine [15], who analysed the M/M/1 queueuing system in which the state of the system is given by (i,j), where 'i' is the number of arrivals and 'j' is the number of departures until time t. This measure provides the better knowledge of a queueing system such as the probability of the exact number of units arrived by time t in the system, the probability of the exact number of units departed by time t from the system, and also provide some other related information. Garg and Kumar [7] obtained explicit time dependent probabilities of exact number of arrivals and departures from the orbit of a single server retrial queue with impatient customers. Kumar and Indra [13] studied the two-state batch departure multiple vacation queueing model.

In this paper, the time dependent probabilities for the exact number of arrivals and departures from the system by a given time when the server is busy and when the server is idle for a single server retrial queueing system with balking is obtained.

The present work has been classified in the following manner: Section 2, gives the formal description of the queueing model. The two-dimensional state model is defined in Section 3. The difference-differential equations are derived and the time dependent solution for the model is obtained. Sections 4 present some interesting measures of effectiveness and provide verification of results. In the Section 5, a computational program using MATLAB software is developed to obtain numerical solution and represented the results are graphically. In the last Section 6, the busy period distribution of the system and the busy period distribution of the server are presented numerically and graphically and the paper ends with conclusion.

2. MODEL DESCRIPTION

We consider a two-state M/M/1 retrial queueing system with balking. A primary customer, who enters the queueing system, arrives according to a Poisson process with rate $\lambda>0$, and its service gets started immediately if the server is idle. On the other hand, if an arriving primary customer finds the server busy, then it either balks the system with probability $(1-\beta)$ or joins the virtual queue (orbit) with probability β. If this customer faces second situation then it retries from the orbit with intensity θ to get service. Service time follows exponential distribution with mean rate μ. For distribution of arrivals, service times and retrials we use the following assumptions and notations:

i. The arrival of primary calls follow a Poisson distribution with parameter λ.

ii. The repeated calls follow a Poisson distribution with parameter θ.

iii. Service times are exponentially distributed with parameter μ.

iv. The stochastic process involved viz. arrivals of units, departures of units and retrials are statistically independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad \text{Re}(s) > 0$$

The Laplace inverse of

$\frac{Q(p)}{P(p)}$ *is* $\sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \frac{Q(p)}{P(p)} (p-a_k)^{m_k}$ $\forall$ p= a_k, $a_i \neq a_k$ for $i \neq k$.

where,

$P(p) = (p-a_1)^{m_1} (p-a_2)^{m_2} \ldots \ldots \ldots (p-a_n)^{m_n}$

Q(p) is a polynomial of degree < $m_1+m_2+m_3+\ldots\ldots\ldots\ldots m_n-1$.

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$, then

$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u) G(t-u) du = F * G$, $F * G$ is called the convolution of F and G.

3. THE TWO-DIMENSIONAL STATE MODEL

3.1 Definitions

$P_{i,j,0}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is idle.

$P_{i,j,1}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t when server is busy.

$P_{i,j}(t)$ = Probability that there are exactly i arrivals in the system and j departures from the system by time t.

$$P_{i,j}(t) = P_{i,j,0}(t) + P_{i,j,1}(t) \quad \forall\ i, j \qquad i \geq j$$

also

$$P_{i,j,1}(t) = 0,\ i \leq j;\ P_{i,j,0}(t) = 0,\ i < j.$$

Initially

$$P_{0,0,0}(0) = 1;\quad P_{i,j,0}(0) = 0\ \&\ P_{i,j,1}(0) = 0,\ i, j \neq 0;$$

3.2 The difference – differential equations governing the system model are:

$$\frac{d}{dt} P_{i,j,0}(t) = -(\lambda + (i-j)\theta) P_{i,j,0}(t) + \mu P_{i,j-1,1}(t) \qquad i \geq j \geq 0 \qquad \textbf{(1)}$$

$$\frac{d}{dt} P_{1,0,1}(t) = -(\lambda\beta + \mu) P_{1,0,1}(t) + \lambda P_{0,0,0}(t) \qquad \textbf{(2)}$$

$$\frac{d}{dt} P_{i,j,1}(t) = -(\lambda\beta+\mu) P_{i,j,1}(t) + \lambda P_{i-1,j,0}(t) + \lambda\beta(1-\delta_{i-1,j}) P_{i-1,j,1}(t) + (i-j)\theta P_{i,j,0}(t)$$

$i>1,\ i>j\geq 0$ **(3)**

where $\delta_{i-1,j} = \begin{cases} 1, & when\ i-1=j \\ 0, & otherwise \end{cases}$

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad Re(s) > 0$$

in the equations (1) - (3) along with the initial conditions, we have

$$(s+\lambda+(i-j)\theta)\bar{P}_{i,j,0}(s) = \mu \bar{P}_{i,j-1,1}(s) \qquad i\geq j\geq 0 \qquad \textbf{(4)}$$

$$(s+\lambda\beta+\mu)\bar{P}_{1,0,1}(s) = \lambda \bar{P}_{0,0,0}(s) \qquad \textbf{(5)}$$

$$(s+\lambda\beta+\mu)\bar{P}_{i,j,1}(s) = \lambda\bar{P}_{i-1,j,0}(s) + \lambda\beta(1-\delta_{i-1,j})\bar{P}_{i-1,j,1}(s) + (i-j)\theta\bar{P}_{i,j,0}(s)$$

$i>1,\ i>j\geq 0$ **(6)**

where $\delta_{i-1,j} = \begin{cases} 1, & when\ i-1=j \\ 0, & otherwise \end{cases}$

3.3 Solution of the Problem:

Solving equations (4) to (6) recursively, we have

$$\bar{P}_{0,0,0}(s) = \frac{1}{s+\lambda} \qquad \textbf{(7)}$$

$$\bar{P}_{1,0,1}(s) = \frac{\lambda}{(s+\lambda)(s+\lambda\beta+\mu)} \qquad \textbf{(8)}$$

$$\bar{P}_{i,0,1}(s) = \frac{(\lambda\beta)^{i-1}}{(s+\lambda\beta+\mu)^{i-1}} \bar{P}_{1,0,1}(s) \qquad i>1 \qquad \textbf{(9)}$$

$$\bar{P}_{1,1,0}(s) = \frac{\lambda\mu}{(s+\lambda)^2(s+\lambda\beta+\mu)} \qquad \textbf{(10)}$$

$$\bar{P}_{i,1,0}(s) = \frac{\mu}{s+\lambda+(i-1)\theta} \bar{P}_{i,0,1}(s) \qquad i>1 \qquad \textbf{(11)}$$

$$\bar{P}_{i,i,0}(s) = \frac{\mu}{s+\lambda}\left(\frac{\lambda}{(s+\lambda\beta+\mu)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{(s+\lambda\beta+\mu)}\bar{P}_{i,i-1,0}(s)\right) \qquad i>1 \qquad \textbf{(12)}$$

$$\bar{P}_{i,i-1,1}(s) = \frac{\lambda}{(s+\lambda\beta+\mu)}\bar{P}_{i-1,i-1,0}(s) + \frac{\theta}{(s+\lambda\beta+\mu)}\bar{P}_{i,i-1,0}(s) \qquad i>1 \qquad \textbf{(13)}$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu}{s+\lambda+(i-j)\theta}\left[\sum_{k=1}^{i-j+1}\left\{\begin{matrix}\left(\frac{1}{(s+\lambda\beta+\mu)}\right)^{(i-j-k+1)}(\lambda)^{\varphi'_k(s)}(\lambda\beta)^{(i-j-k)\varphi'_k(s)}\eta'_k(s) \\ \bar{P}_{j+k-1,j-1,0}(s)\end{matrix}\right\} + \left\{\frac{(\lambda\beta)^{(i-j)}}{(s+\lambda\beta+\mu)^{(i-j)}}\bar{P}_{j,j-1,1}(s)\right\}\right] \qquad i>j>1 \qquad \textbf{(14)}$$

where $\eta'_k(s) = \begin{cases} 1 & for\ k=1 \\ \left(1+\frac{k\theta\beta}{(s+\lambda\beta+\mu)}\right) & for\ k=2\ to\ i-j \\ \frac{k\theta}{(s+\lambda\beta+\mu)} & for\ k=i-j+1 \end{cases}$

ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 6, Issue 3, pp: (528-539), Month: July - September 2018, Available at: www.researchpublish.com

and $\varphi'_k(s) = \begin{cases} 1 & \text{for } k = 1 \text{ to } i-j-1 \\ 0 & \text{for } \quad k = i-j \end{cases}$

$$\bar{P}_{i,j,1}(s) = \sum_{k=1}^{i-j}\left\{\left(\frac{1}{(s+\lambda\beta+\mu)}\right)^{(i-j-k)}(\lambda)^{\varphi'_k(s)}(\lambda\beta)^{(i-j-k-1)\varphi'_k(s)}\eta'_k(s)\bar{P}_{j+k,j,0}(s)\right\} + \left\{\frac{(\lambda\beta)^{(i-j-1)}}{(s+\lambda\beta+\mu)^{(i-j-1)}}\bar{P}_{j,j-1,1}(s)\right\}$$

$i \geq j+2,\ j \geq 1$ **(15)**

where $\eta'_k(s) = \begin{cases} 1 & for\ k = 1 \\ \left(1+\frac{k\theta\beta}{(s+\lambda\beta+\mu)}\right) & for\ k = 2\ to\ i-j-1 \\ \frac{k\theta}{(s+\lambda\beta+\mu)} & for\ k = i-j \end{cases}$

and $\varphi'_k(s) = \begin{cases} 1 & \text{for } k = 1 \text{ to } i-j-1 \\ 0 & \text{for } \quad k = i-j \end{cases}$

Taking the Inverse Laplace transform of equations (7) to (15), we have

$$P_{0,0,0}(t) = e^{-\lambda t} \quad \textbf{(16)}$$

$$P_{1,0,1}(t) = \lambda e^{-\lambda t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\left(\frac{\mu}{\beta}\right)}\right\} \quad \textbf{(17)}$$

$$P_{i,0,1}(t) = (\lambda\beta)^{i-1}e^{-(\lambda\beta+\mu)t}\frac{t^{i-2}}{(i-2)!} * P_{1,0,1}(t) \qquad i > 1 \quad \textbf{(18)}$$

$$P_{i,1,0}(t) = \mu e^{-(\lambda+(i-1)\theta)t} * P_{i,0,1}(t) \qquad i \geq 1 \quad \textbf{(19)}$$

$$P_{i,i,0}(t) = \left[(\lambda\mu)e^{-\lambda t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\left(\frac{\mu}{\beta}\right)}\right\} * P_{i-1,i-1,0}(t) + (\mu\theta)e^{-\lambda t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\left(\frac{\mu}{\beta}\right)}\right\} * P_{i,i-1,0}(t)\right]$$

$i \geq 1$ **(20)**

$$P_{i,i-1,1}(t) = \left(\lambda e^{-(\lambda\beta+\mu)t} * P_{i-1,i-1,0}(t) + \theta e^{-(\lambda\beta+\mu)t} * P_{i,i-1,0}(t)\right) \qquad i > 1 \quad \textbf{(21)}$$

$$P_{i,j,1}(t) = \lambda^{i-j-1}(\beta)^{i-j-2}e^{-(\lambda\beta+\mu)t}\frac{t^{i-j-2}}{(i-j-2)!} * P_{j+1,j,0}(t) + \sum_{k=2}^{i-j-1}\left[\lambda^{i-j-k}(\beta)^{i-j-k-1}e^{-(\lambda\beta+\mu)t}\frac{t^{i-j-k-1}}{(i-j-k-1)!} * P_{j+k,j,0}(t)\right] + \sum_{k=2}^{i-j-1}\left[\lambda^{i-j-k}(\beta)^{i-j-k-1}(k\theta\beta)e^{-(\lambda\beta+\mu)t}\frac{t^{i-j-k}}{(i-j-k)!} * P_{j+k,j,0}(t)\right] + (i-j)\ \theta e^{-(\lambda\beta+\mu)t} * P_{i,j,0}(t) + (\lambda\beta)^{i-j-1}e^{-(\lambda\beta+\mu)t}\frac{t^{i-j-2}}{(i-j-2)!} * P_{j+1,j,1}(t)$$

$i \geq j+2,\ j \geq 1$ **(22)**

$$P_{i,j,0}(t) = \mu\lambda^{i-j}(\beta)^{i-j-1}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j-1,0}(t) + \mu e^{-(\lambda+(i-j)\theta)t}\left[\sum_{k=2}^{i-j}(\lambda)^{i-j-k+1}(\beta)^{i-j-k}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+1}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+1-r}}\right\} * P_{j+k-1,j-1,0}(t)\right] + \mu e^{-(\lambda+(i-j)\theta)t}\left[\sum_{k=2}^{i-j}(\lambda)^{i-j-k+1}(\beta)^{i-j-k}(k\theta\beta)\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+2}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-k+1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-k+2-r}}\right\} * P_{j+k-1,j-1,0}(t)\right] + \mu(i-j+1)\theta e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)} - \frac{e^{-\left(\frac{\mu}{\beta}\right)t}}{\left(\frac{\mu}{\beta}\right)}\right\}P_{i,j-1,0}(t) + \mu(\lambda\beta)^{i-j}e^{-(\lambda+(i-j)\theta)t}\left\{\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j}} - e^{-\left(\frac{\mu}{\beta}\right)t}\sum_{r=0}^{i-j-1}\frac{t^r}{r!}\frac{1}{\left(\frac{\mu}{\beta}\right)^{i-j-r}}\right\} * P_{j,j-1,1}(t)$$

$i > j > 1$ **(23)**

ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
Vol. 6, Issue 3, pp: (528-539), Month: July - September 2018, Available at: www.researchpublish.com

4. SOME IMPORTANT PERFORMANCE MEASURES OF THE MODEL AND VERIFICATION OF RESULTS

4.1 The Laplace transform of $\bar{P}_{i.}(s)$ of the probability $P_{i.}(t)$ that exactly i units arrive by time t is:

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i} \bar{P}_{i,j}(s) = \frac{\lambda^i}{(s+\lambda)^{i+1}}; \qquad i > 0 \tag{24}$$

And its Inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda t}(\lambda t)^i}{i!}. \tag{25}$$

The assumption on primary arrivals is that it forms a Poisson process and analysis of above abstract solution also verifies the same.

4.2 The departure process from M/M/1 queue has the distribution function $P_{.j}(t)$, the probability that exactly j customers have been served by time t. $P_{.j}(t)$ in terms of $P_{i,j}(t)$ is given by:

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t)$$

4.3 From the solution of our model, we verify that the sum of all possible probabilities is one i.e. taking summation over i and j on equations (7) - (15) and adding, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{\bar{P}_{i,j,0}(s) + \bar{P}_{i,j,1}(s)\} = \frac{1}{s}.$$

After taking the inverse Laplace transformation, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\{P_{i,j,0}(t) + P_{i,j,1}(t)\} = 1.$$

which is a verification of our results.

4.4 Define $Q_{n,k}(t)$ = Probability that there are n customers in the system at time t and the server is free or busy according as k=0 or 1.

The probability of exactly n customers in the system at time t in terms of $P_{i,j,0}(t)$ and $P_{i,j,1}(t)$:

When the server is free, it is defined by probability $Q_{n,0}(t)$:

$$Q_{n,0}(t) = \sum_{j=0}^{\infty} P_{j+n,j,0}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures).

When the server is busy, it is defined by probability $Q_{n,1}(t)$:

$$Q_{n,1}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j,1}(t)$$

In this case, the number of customers in the orbit is equal to n which is obtained by using:

n = (number of arrivals – number of departures – 1).

Using above definitions, from the equations (1) to (3), the set of equations in statistical equilibrium are:

$(\lambda + n\theta)\, Q_{n,0} = \mu\, Q_{n,1}$ $\quad n \geq 0$ **(26)**

$(\lambda\beta + \mu)\, Q_{n,1} = \lambda Q_{n,0} + \lambda\beta Q_{n-1,1} + (n+1)\,\theta\, Q_{n+1,0}$ $\quad n > 1$ **(27)**

These equations coincide with the results (3.68) of Falin & Templeton (1997).

5. NUMERICAL SOLUTION

In order to show the effect of different parameters on our model, some numerical results are generated using MATLAB programming. These numerical results are obtained for these arbitrary values of $\rho=\left(\frac{\lambda}{\mu}\right)$, $\eta=\left(\frac{\theta}{\mu}\right)$ and balking parameter (1-β) which satisfy the stability condition. From the numerical results, it is found that the sum of all the probabilities at any instance approaches to one. In table 1, we show some of the significant probabilities at different instants of time and their sum (taking $\rho=\left(\frac{\lambda}{\mu}\right) = 0.3$, $\eta=\left(\frac{\theta}{\mu}\right) = 0.6$ and (1-β) = 0.4).

Table-1: Some significant probabilities of System state at different instant of time

At time t= 1

$P_{0,0,0}$	$P_{1,1,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	Sum
0.7408	0.0846	0.1478	0.0114	0.0078	0.9924

At time t= 5

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,1,1}$
0.2231	0.295	0.019	0.1439	0.0159	0.0358	0.0751	0.0145	0.0829	0.0181

$P_{3,2,1}$	$P_{4,3,1}$	Sum
0.0336	0.0079	0.9648

At time t= 10

$P_{0,0,0}$	$P_{1,1,0}$	$P_{2,2,0}$	$P_{3,2,0}$	$P_{3,3,0}$	$P_{4,3,0}$	$P_{4,4,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$
0.0498	0.1504	0.2	0.0149	0.1555	0.0169	0.0834	0.0111	0.0388	0.0077

$P_{1,0,1}$	$P_{2,0,1}$	$P_{2,1,1}$	$P_{3,1,1}$	$P_{3,2,1}$	$P_{4,2,1}$	$P_{4,3,1}$	$P_{5,3,1}$	$P_{5,4,1}$	Sum
0.017	0.0035	0.0485	0.0133	0.0616	0.0157	0.0456	0.012	0.0217	0.9674

At time t= 20

$P_{0,0,0}$	$P_{2,2,0}$	$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,4,0}$	$P_{5,5,0}$	$P_{6,5,0}$	$P_{6,6,0}$	$P_{7,6,0}$	$P_{7,7,0}$
0.0159	0.0486	0.0994	0.1317	0.0093	0.1392	0.0012	0.1072	0.0117	0.0707

ISSN 2348-1218 (print)
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$P_{8,7,0}$	$P_{8,8,0}$	$P_{9,9,0}$	$P_{10,10,0}$	$P_{2,1,1}$	$P_{3,2,1}$	$P_{4,3,1}$	$P_{5,3,1}$	$P_{5,4,1}$	$P_{6,4,1}$
0.0089	0.0386	0.0178	0.0139	0.0056	0.0169	0.0318	0.0056	0.0419	0.0078

$P_{6,5,1}$	$P_{7,5,1}$	$P_{7,6,1}$	$P_{8,6,1}$	$P_{8,7,1}$	$P_{9,8,1}$	$P_{10,9,1}$	Sum
0.0415	0.0080	0.0323	0.0064	0.0203	0.0106	0.0057	0.9593

At time t= 30

$P_{3,3,0}$	$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,7,0}$	$P_{8,8,0}$	$P_{9,8,0}$	$P_{9,9,0}$	$P_{10,9,0}$
0.0177	0.0393	0.0675	0.0864	0.1143	0.0069	0.1033	0.0098	0.0858	0.0176

$P_{10,10,0}$	$P_{4,3,1}$	$P_{5,4,1}$	$P_{6,5,1}$	$P_{7,6,1}$	$P_{8,6,1}$	$P_{8,7,1}$	$P_{9,7,1}$	$P_{9,8,1}$	$P_{10,8,1}$
0.2478	0.0063	0.0138	0.0232	0.0313	0.0058	0.0246	0.0065	0.0323	0.0074

$P_{10,9,1}$	Sum
0.0322	0.9798

At time t= 40

$P_{4,4,0}$	$P_{5,5,0}$	$P_{6,6,0}$	$P_{7,7,0}$	$P_{8,8,0}$	$P_{9,9,0}$	$P_{10,9,0}$	$P_{10,10,0}$	$P_{7,6,1}$	$P_{8,7,1}$
0.0068	0.0181	0.0354	0.0575	0.0794	0.0946	0.0099	0.5718	0.0113	0.0181

$P_{9,8,1}$	$P_{10,8,1}$	$P_{10,9,1}$	Sum
0.0223	0.0073	0.0363	0.9688

Through figures 1 to 7, various probabilities are graphically represented against time. In figure 1, the probabilities $P_{0,0,0}$ and $P_{1,1,0}$ are plotted against time t for the case when $\rho=\left(\frac{\lambda}{\mu}\right)=0.6$, $\eta=\left(\frac{\theta}{\mu}\right)=0.9$, $(1-\beta)=0.7$. It is clear from the graph that the probability $P_{0,0,0}$ decreases rapidly from the initial value 1. The probability $P_{1,1,0}$ increases rapidly in the starting moments from initial value zero, and then decreases gradually.

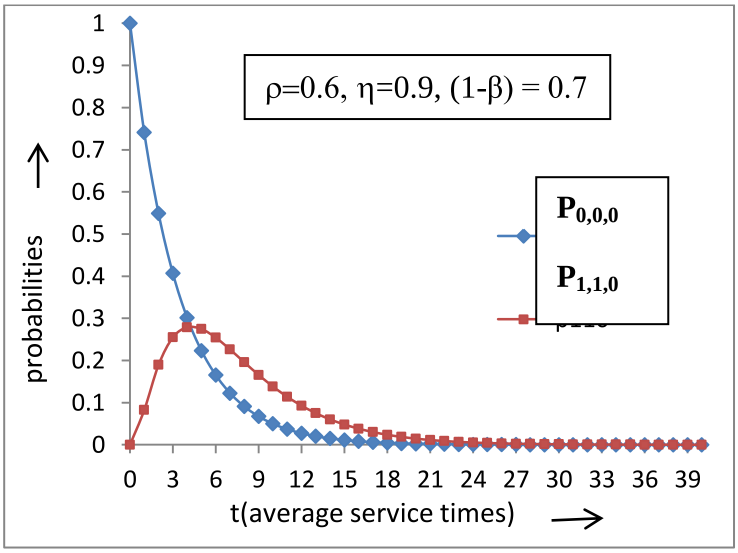

Fig. 1: Probabilities $P_{0,0,0}$ and $P_{1,1,0}$ against time t

ISSN 2348-1218 (print)
International Journal of Interdisciplinary Research and Innovations ISSN 2348-1226 (online)
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Figure 2 shows relative change among four probabilities i.e. $P_{4,0,1}$, $P_{4,1,1}$, $P_{4,2,1}$ and $P_{4,3,1}$ against time for the case when ρ=0.3, η=0.6 and (1-β) =0.4. From this graph, we interpret that for higher values of time, the probabilities follow the rule that higher the number of departures, higher is the probability as it is observed here that $P_{4,3,1} > P_{4,2,1} > P_{4,1,1} > P_{4,0,1}$.

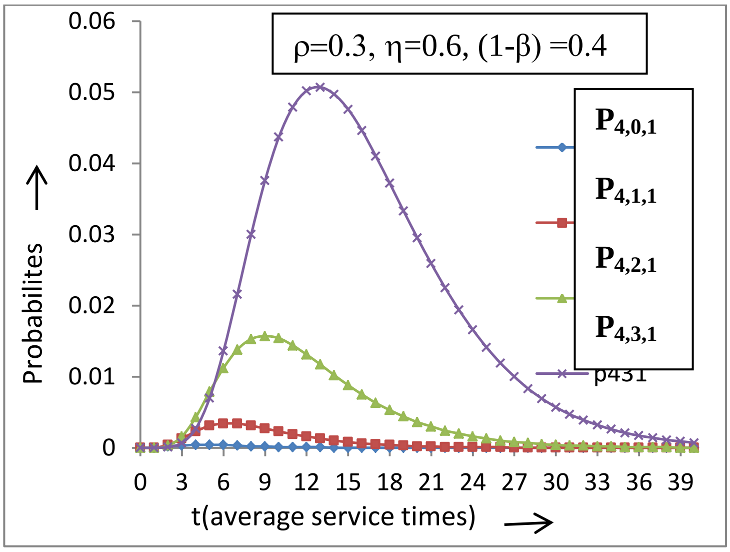

Fig. 2: Probabilities $P_{4,0,1}$, $P_{4,1,1}$, $P_{4,2,1}$ and $P_{4,3,1}$ against time t

The comparison among four probabilities i.e. $P_{3,1,1}$, $P_{4,1,1}$, $P_{5,1,1}$ and $P_{6,1,1}$ are shown in figure 3 for the case when ρ=0.6, η=0.6, (1-β) =0.1. The probabilities increases in the starting moments but afterwards start decreasing due to their special behavior. It can also be interpreted for the case under study that probabilities $P_{i,1,1}$ have smaller values when i (the number of arrivals) are more.

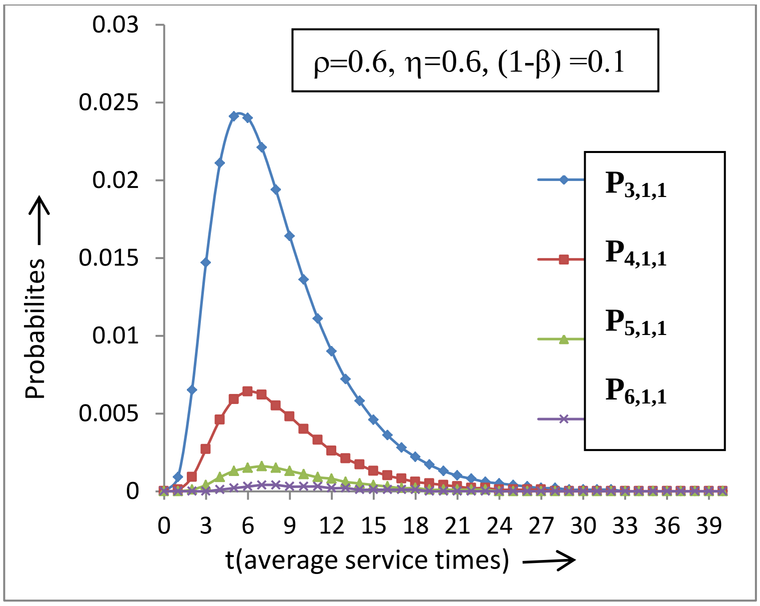

Fig. 3: Probabilities $P_{3,1,1}$, $P_{4,1,1}$, $P_{5,1,1}$ and $P_{6,1,1}$ against time t

The plot of probabilities $P_{4,1,0}$ and $P_{4,1,1}$ against time t is shown in figure 4 for the case when ρ=0.6, η=0.9, (1-β) =0.4. The Probabilities $P_{4,1,1}$ and $P_{4,1,0}$ increased rapidly in the starting moment and then decreases with a high rate. It is seen that probability $P_{4,1,1}$ always remained more than $P_{4,1,0}$ which shows that when there are waiting units in the system, the probability when the server is busy is always more than the probability when the server is free.

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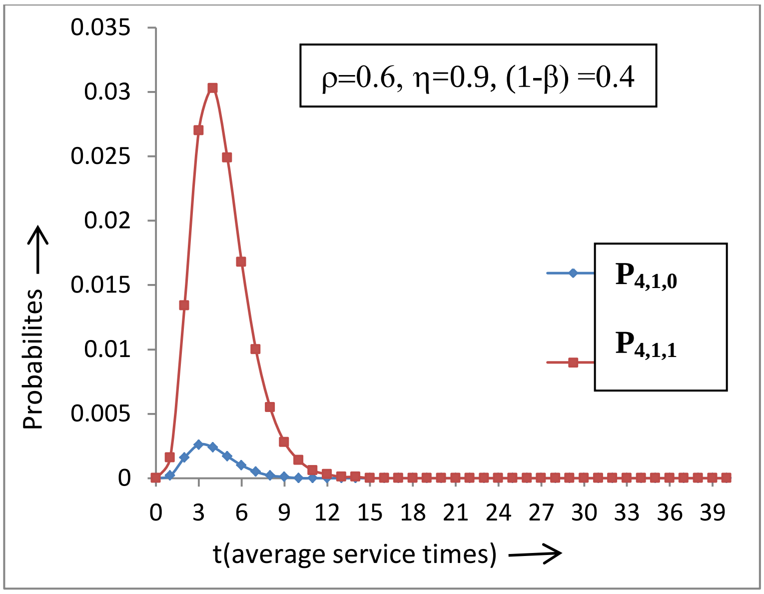

Fig. 4: Probabilities $P_{4,1,0}$ and $P_{4,1,1}$ vs. time t

To study the effect of traffic intensity on different probabilities of the model, the data of various probabilities is generated for different values of ρ keeping the other parameters constant. The set of ρ values for the comparison is {0.3,0.6,0.9}. Figure 5 depicts plot of probability $P_{0,0,0}$ against time t for different value of ρ. From the initial condition we see that the probability $P_{0,0,0}$ at time t=0 is one for all ρ and start decreasing with time. Behavior of the probability $P_{0,0,0}$ is same for all the values of ρ. From the figure, it is concluded that as ρ increases $P_{0,0,0}$ decreases. So more the traffic intensity i.e. more customers are arriving per unit service time, less is the probability of zero units in the system.

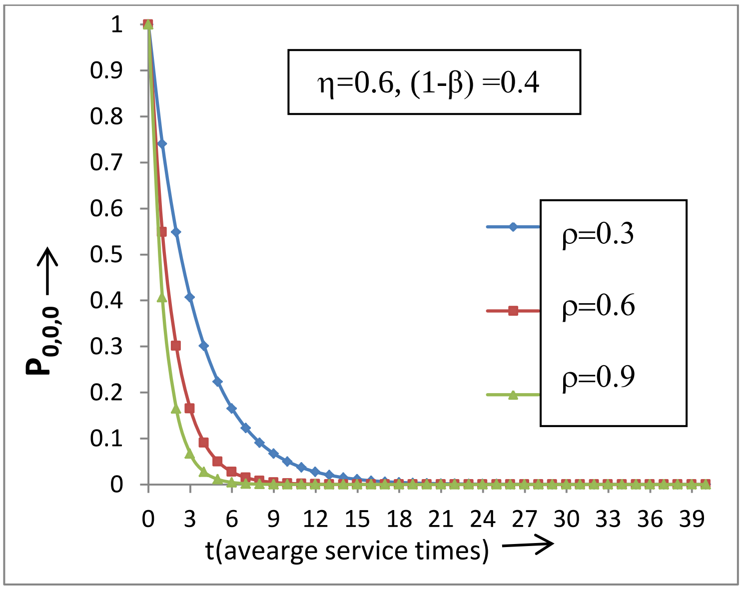

Fig. 5: Effect of ρ=[λ/μ] on $P_{0,0,0}$ against time

To compare the models with balking and without balking, some of the probabilities are plotted in figures 6 and 7. Here, it is observed that the gap between various values of probabilities $P_{2,1,1}$ and $P_{3,1,1}$ is much more in case of balking as compared to the case of without balking. Similarly it is found that the gap between the various values of probabilities $P_{2,1,1}$ and $P_{4,1,1}$ in case of model with balking is much more as compared to the model without balking. These results are the expected ones as in balking model there are less chances of going to state $P_{3,1,1}$ and then to $P_{4,1,1}$ from state $P_{2,1,1}$ as compared to model without balking.

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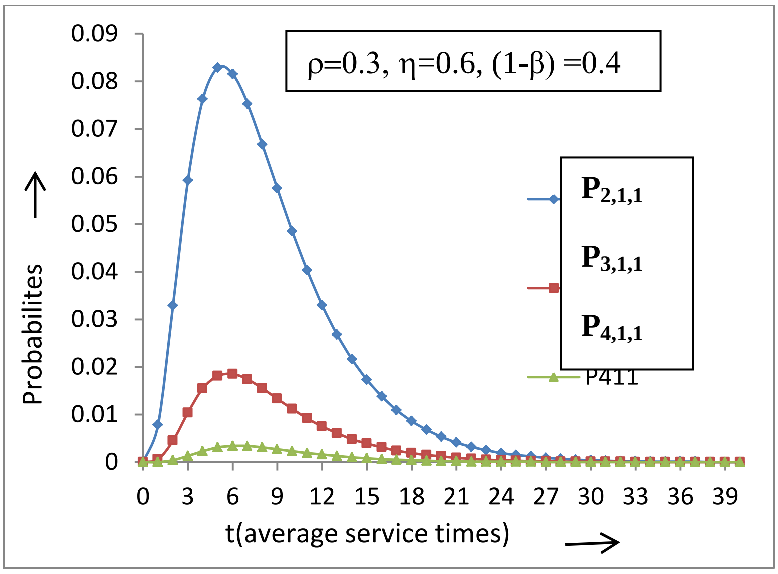

Fig. 6: Various probabilities against time in model with balking

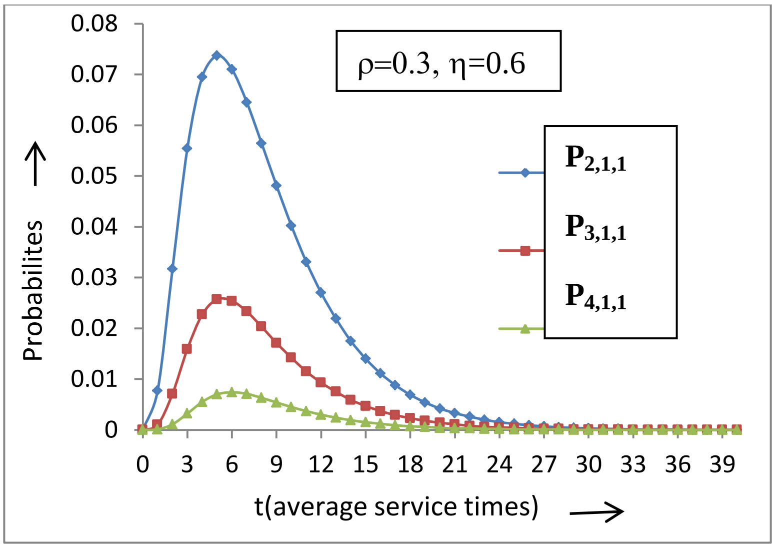

Fig. 7: Various probabilities against time in model without balking

6. BUSY PERIOD DISTRIBUTION

In this section, some appealing numerical results about busy period distribution of the server and the system are obtained.

The probability when the server is busy is given by

P (Server is busy) = $\sum_{i>j\geq 0} P_{i,j,1}$ (t) **(29)**

The probability when the system is busy is given by

P (System is busy) = $\sum_{i>j\geq 0}\left(P_{i,j,0}\,(t) + P_{i,j,1}\,(t)\right)$ **(30)**

Numerical & Graphical representation of busy period:

The numerical results for when the system is busy and for when the server is busy are obtained using MATLAB software. The probabilities for different values of ρ (traffic intensity) at η=0.6 and (1-β) = 0.4 for when the system is busy and for when the server is busy are presented in table-2.

Table-2: Probability of system busy and server busy for the case when η=0.6 at different ρ.

	Probability (System busy)			Probability (Server busy)		
t	ρ=0.3	ρ=0.6	ρ=0.9	ρ=0.3	ρ=0.6	ρ=0.9
0	0	0	0	0	0	0
1	0.1725	0.3147	0.4323	0.1684	0.3012	0.4065
2	0.2332	0.4183	0.5626	0.2176	0.3716	0.4833
3	0.2636	0.4756	0.6357	0.2356	0.3979	0.5132
4	0.2828	0.5152	0.6858	0.2447	0.4139	0.5333
5	0.2964	0.5445	0.7223	0.2506	0.4257	0.549
6	0.3062	0.567	0.7498	0.2548	0.4351	0.5617
7	0.3136	0.5845	0.7711	0.2578	0.4425	0.5717

Different probabilities are plotted against time through figures 8 to 10. The relationship between the probability (server busy) and the probability (system busy) for the case ρ=0.9, η=0.6 and (1-β) = 0.4 are shown in figure 8. From this figure, it is apparent that the two probabilities are increasing rapidly in starting moments and they start decreasing gradually for higher values of t. we can also interpret that the probability when the system is busy always remains more than the probability when the server is busy.

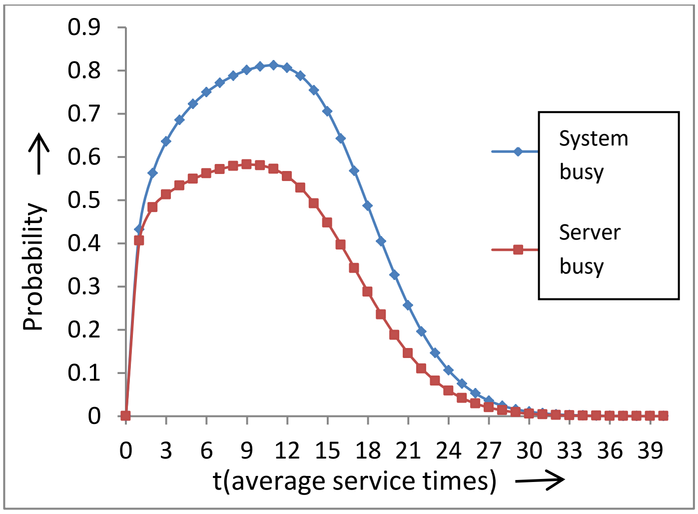

Fig. 8: Probability (system busy) and Probability (server busy) against time for ρ=0.9, η=0.6

In figure 9, the probability (system busy) and in figure 10, the probability (server busy) are plotted for different values of ρ (traffic intensity) and taking (1-β) = 0.4. From these figures it is examined that as the value of ρ increases, the probabilities (system busy) and (server busy) achieved higher highest values for some t.

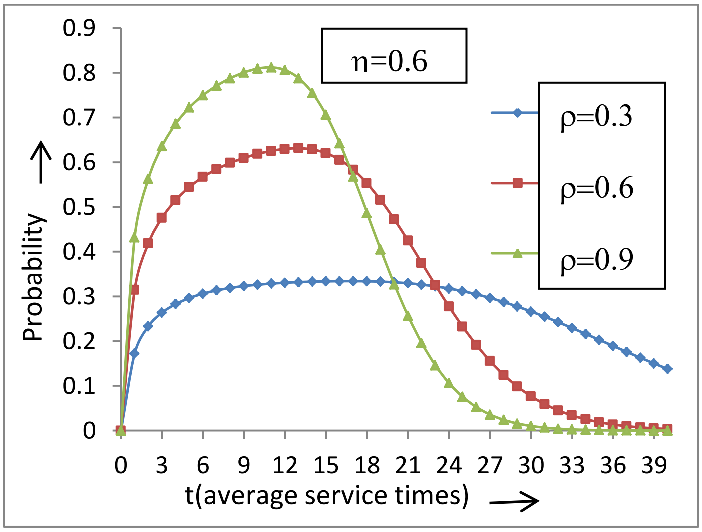

Fig. 9: Effect of ρ on Probability (system busy) against time

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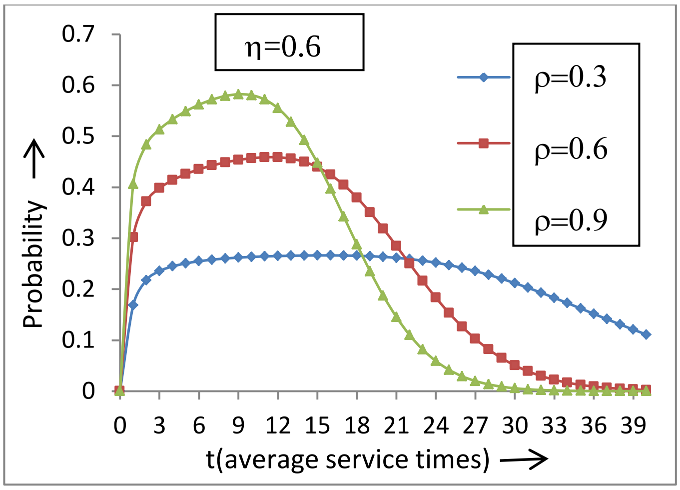

Fig. 10: Effect of ρ on Probability (server busy) against time

7. CONCLUSION

In this paper, a single server retrial queueing system with repeated attempts and balking is considered. Factors are well understood and quantified from two-dimensional state queueing model. For this model, time dependent probabilities for exact numbers of arrivals and exact number of departures for when the system is busy and when the system is free are evaluated. Various performance measures and special cases have also been analysed. The effects of various parameters on our two-dimensional state retrial queueing model are illustrated numerically and graphically.

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